# Outline

Three major transitions in Peirce's development of the quantification theory are identified: the *indexical*, which presupposed a substitutional interpretation, the *symbolic* (objectual/game-theoretic interpretation), and the *iconic* (diagrammatic/continuity) interpretation. These interpretations establish the senses in which Peirce meant logic to be the science of semeiotic.

1. The development of quantifiers
2. The indexical, symbolic and iconic transitions
3. Substitutional vs. objectual interpretations.

> *Peirce's theory of signs, or semeiotic, misunderstood by so many, has gotten in amongst the wrong crowd. It has been taken up by an interdisciplinary army of 'semioticians' whose views and aims are antithetical to Peirce's own, and meanwhile it has been shunned by those philosophers who are working in Peirce's own spirit on the very problems to which his semeiotic was addressed. (Tom Short)*

# The Algebraic Tradition

George Boole's (1815–1864) algebra (Boolean algebra) in 1847 consisted of the operations of

- Product (intersection) "," $(x, y)$
- Sum (union) "+" $(x + y)$
- Complementation "$^c$" $(x^c)$

Augustus De Morgan (1806–1871) suggested to consider (two-place) relations in logic, and took Boolean algebra to form the basis for logical operations.

# The Early Calculus of Relations (1867)

Peirce followed suit and regarded Boolean algebra as a propositional calculus:

- *Product* $x, y$ is regarded as a *conjunction*
- *Sum* $x + y$ is regarded as an inclusive *disjunction*
- *Complementation* $x^c$ is regarded as a *negation*.

He took two-place relations to be *sets of ordered pairs* $(a, b)$ ("dyads"), three-place relations *sets of ordered triples* $(a, b, c)$ ("triads"), and relations generally as sets of *ordered n-tuples* of singular objects.

# The Early Calculus of Relations (1867)

An extension of Boole's logic to operations on relations:

1. The relative terms *being a lover of* ($L$) and *being a benefactor of* ($B$) give rise to a **product**

$$L, B \quad (\textit{is a lover and a benefactor of}) \tag{1}$$

just in case there is an ordered pair $(a, b)$ of objects $a$ and $b$ standing in both of the relations $L$ and $B$.

2. *a lover of* ($L$) and *a benefactor of* ($B$) give rise to a **sum**

$$L + B \quad (\textit{is a lover or a benefactor of}) \tag{2}$$

just in case there is an ordered pair $(a, b)$ standing in one or both of the relations $L$ and $B$.

3. *a lover of* ($L$) has a **complementation**

$$L^c \quad (\textit{is not a lover of}) \tag{3}$$

just in case there is $(a, b)$ which does not stand in the relation $L$...

# The Logic of Relatives (1870)

The additional signs are the *relative* products and sums:

1. Concatenation of two relative terms $L$ and $B$ gives rise to a **relative product**:

$$LB \quad (\textit{is a lover of a benefactor of}) \tag{4}$$

   just in case there is an ordered pair $(a, b)$ standing in the relation in which $a$ is a lover of some $c$ and $c$ is a benefactor of $b$ (that is, the first object $a$ is a lover of a benefactor of the second object $b$).

2. (1883) Relation between $L$ and $B$ gives rise to a **relative sum**:

$$L \dagger B \quad (\textit{is a lover of every benefactor of}) \tag{5}$$

   just in case there is an ordered pair $(a, b)$ standing in the relation in which $a$ is the lover of every $c$ who is the benefactor of $b$.

So the crucial difference was between asserting something about *particulars* (*some*, *a few*, *not all*) and *generals* (*every*, *all*, *any*).

# The Logic of Relatives (1870)

Peirce also defined the relation of

3. **Hypotheticals**: $\prec$ (class inclusion) is a relation between two relative terms, $L$ and $B$, $L \prec B$, denoting that all lovers (of anything or anyone) are benefactors (of that same thing).

We tend nowadays to use the arrow $\varphi \to \psi$, for a (material) *conditional* between *propositions*.

# Universes of Discourse

We need something else still, namely the *universe of discourse*. The idea and the term comes from De Morgan (1846):

> *Writers on logic, it is true, do not find elbow-room enough in anything less than the whole universe of possible conceptions; but the universe of a particular assertion or argument may be limited in any matter expressed or understood. . . By not dwelling on this power of making what we may properly. . . call the universe of a proposition, or of a name, matter of expressive definition, all rules remaining the same, writers on logic deprive themselves of much useful illustration.*

According to Peirce, this date marked the birth of 'exact logic' (MS 450).

# Universes of Discourse

Peirce proposed

> *to use the term "universe" to denote that class of individuals about which alone the whole discourse is understood to run. The universe, therefore, in this sense, as in Mr. De Morgan's, is different on different occasions. (CP 3.65, 1870)*

Besides individuals, Peirce had collections, qualities, modalities etc. in the universes of discourse.

He made the use of such universes a thoroughly logical issue, and applied it to his logic of relatives.

> *A logical universe is, no doubt, a collection of logical subjects, but not necessarily of meta-physical Subjects, or 'substances'. (CP 4.546, 1906)*

# 1885, Quantifiers

Let $a_1, a_2, a_3, \ldots$ be all individual objects of the given universe of discourse, and let the *value* of term $F$ for $a_i$ be

$$[F]_i = 1 \text{ if and only if } a_i \text{ is } F, \text{ otherwise } [F]_i = 0.$$

Likewise $L$, as applied to $(a_i, a_j)$, has values

$$[L]_{ij} = 1 \text{ iff } a_i \text{ loves } a_j, \text{ and } [L]_{ij} = 0 \text{ otherwise.}$$

Saying that *some* individual is $F$ is true iff

$$\Sigma_i \, [F]_i > 0. \tag{6}$$

Saying that *every* individual is $F$ is true iff

$$\Pi_i \, [F]_i > 0. \tag{7}$$

# Quantifiers

*Something loves something* is now expressed by

$$\Sigma_i \Sigma_j \, [L]_{ij} > 0. \tag{8}$$

*Everything is a lover of something* is expressed by

$$\Pi_i \Sigma_j \, [L]_{ij} > 0. \tag{9}$$

Arbitrarily iterating the two operators we can express very complex things. . .

# Quantifiers

. . . just by applying strings of operators to matrices:

> *Any proposition whatever is equivalent to saying that some complexus of aggregates [sums] and products of numerical coefficients is greater than zero. Thus,*
>
> $$\Sigma_i \Sigma_j L_{ij} > 0 \tag{10}$$
>
> *means that something is a lover of something; and*
>
> $$\Pi_i \Sigma_j L_{ij} > 0 \tag{11}$$
>
> *means that everything is a lover of something. (CP 3.351)*

This has a snag, however. . .

# Quantifiers

For the universes of discourse may well be uncountable. Thus operators should not be simply sums and products of objects:

> *In order to render the notation as iconical as possible we may use* $\Sigma$ *for* **some**, *suggesting a sum [disjunction], and* $\Pi$ *for* **all**, *suggesting a product [conjunction]. . . . It is to be remarked that* $\Sigma_i x_i$ *and* $\Pi_i x_i$ *are only similar to a sum and product; they are not strictly of that nature, because the individuals of the universe may be innumerable. (CP 3.393, 1885)*

- Sums not well-defined for uncountable pairs of indices
- First-order logic doesn't differentiate countable from uncountable universes (cf. Löwenheim 1915).

[The assimilation of quantifying operators and infinitely long sequences of connectives was later said by Wittgenstein to have been the biggest mistake he made in the *Tractatus*.]

# Quantifiers

Accordingly, Peirce simplified the notation:

- $\Sigma_i F_i$ (*Something, $i$, is $F$*)
- $\Pi_i F_i$ (*Everything, $i$, is $F$*)
- $\Pi_i \Sigma_j L_{ij}$ (*Everybody, $i$, loves somebody, $j$*)
- ...

These are the quantifiers proper, in the Peano notation

- $\exists x\, F(x)$, the 'existential' quantifier
- $\forall x\, F(x)$, the universal quantifier
- $\forall x \exists y\, L(x, y)$
- ...

# Quantifiers

Operations on relations are thus defined as follows:

- Relative product: $(LB)_{ik} ::= \Sigma_j (L_{ij} B_{jk})$ (*is a lover of a benefactor of*)
- Relative sum: $(L \dagger B)_{ik} ::= \Pi_j (L_{ij} + B_{jk})$ (*is a lover of every benefactor of*)

In the contemporary notation, these are

- $\exists y \, (L(x, y) \wedge B(y, z))$ (*There exists/Someone, $y$, is such that...*)
- $\forall y \, (L(x, y) \vee B(y, z))$ (*For all $y$; Anything, $y$, is such that...*)

Thus emerged first-order logic!
("First-intentional logic of relatives").
(Note the free indices/variables and the prenex normal form.)

# First vs. Second-intentional Logic

Peirce differentiated between quantifying *individual objects* ("first-intentional") and quantifying *relations* ("second-intentional" logic).

For example, the identity relation $I_{ij}$ is defined by the second-order formula

$$I_{ij} ::= \Pi_X((X_i X_j) + (X_i^c X_j^c)). \tag{12}$$

This is Leibniz' Principle of the *Identity of Indiscernibles*.

> **First intentions** *are those concepts which are derived by comparing percepts, such as ordinary concepts of classes, relations, etc.* **Second intentions** *are those which are formed by observing and comparing first intentions. (CP 2.548)*

# 1885: Quantification based on Indices

- Relatives have indices $i, j, k, \ldots$
- Indices 'refer directly'; we can recognise the actual universe of discourse only if we can pick its objects directly:

*The [indexical] sign signifies its object solely by virtue of being really connected with it. . . . the subscript numbers in algebra distinguish one value from another without saying what those values are. (3.361, Peirce 1885: 164)*

- Objects are named by indices
- Individual variables of quantifiers are indices
- Linguistic significance of quantification (algebra $\rightsquigarrow$ language):

*The index asserts nothing; it only says "There!" It takes hold of our eyes, as it were, and forcibly directs them to a particular object, and there it stops. Demonstrative and relative pronouns are nearly pure indices, because they denote things without describing them.(3.361)*

# 1885: Substitutional interpretation

1885 presupposes a substitutional interpretation of quantification: Directly referring indices connect language with the objects of the domains.

However, substitutional interpretation is bound to fail:

1. "suggesting a sum...suggesting a product": uncountability
2. Different quantifier orderings have different substitutional equations: (13) is valid but (14) "does not hold when the $i$ and $j$ are not separated" (p. 231):

$$\Sigma_i \Pi_j x_{ij} \succ \Pi_j \Sigma_i x_{ij} \tag{13}$$

$$\Pi_j \Sigma_i x_{ij} \succ \Sigma_i \Pi_j x_{ij} \tag{14}$$

$\Longrightarrow$ Inferential relationships presuppose certain quantifier orderings, which are presupposed by the substitutional equations

3. No special class of substitution instances available in language.

$\Longrightarrow$ Quantifiers are not indexical signs after all.

# Post-1885: Objectual Interpretation

So Peirce gives up 'logic', but he does not give up 'semantics'.

- Variables do not refer directly:

  [Π and Σ] *show whether the individuals are to be selected universally or existentially, that is, by the interpreter or by the utterer. (MS L 107: 8, 1905).*
  *In the sentence "Every man dies," "Every man" implies that the interpreter is at liberty to pick out a man and consider the proposition as applying to him. (CP 5.542, c.1902).*

- Logical constants interpreted by *habits* (strategies):
  - An objectual, game-theoretic interpretation
  - Normative logic based on 'commitment rules' concerning the truth at terminal points of plays of the games
  - Variable–individual connections mediated by "habits" in the "quasi-minds" of the utterer and the interpreter
  - Habits connect language with the individuals of domains.

$\Longrightarrow$ Logical constants must be symbols.

# The Iconic Turn, 1895–

The third transition:

- Logical constants are icons ('express their own meaning')
- Notational unification for logical constants (Logic of Existential Graphs)
- Semantics by habits $\rightsquigarrow$ objectual interpretation preserved
- Continuous domains
  - Substitutional interpretation fails
  - Frege–Russell thesis fails
  - Inferentialism fails (cannot get the TONK)
- Higher-dimensional algebras?

# Conclusions

Summarising, the Theory of Quantification (1885) emerged out of

- Developments on the algebraic calculus of relatives (1870) grounded in Boolean algebra and De Morgan's idea of universes of discourse.

Why *existential* quantification?

- Universal assertions *every*, *all*, *any*, used to denote non-existence of exceptions in the universe of discourse. Hence, particular assertions evolved into existential quantifiers by way of duality considerations:
  - The meaning of a relative sum: there is a relation $(a, b)$ in some universes such that no exceptions to that relation may exist.
  - So a particular quantifier would better denote existence.

# Conclusions

1. Beginning 1885, Peirce discontinues development of the quantification theory. Why?
   - He sees that the substitutional interpretation ekes out logic?
   - Proper names not 'rigid designators'?
2. Logic *is* formal semeiotic.
3. The wider goals:

   *I even hope that what I have done may prove a first step toward the resolution of one of the main problems of logic, that of producing a method for the discovery of methods in mathematics. (1885: 166).*

   *The calculus of the new logic, which is applicable to everything, will certainly be applied to settle certain logical questions of extreme difficulty relating to the foundations of mathematics. Whether or not it can lead to any method of discovering methods in mathematics it is difficult to say. Such a thing is conceivable. (3.454, Review of Schröder, 1896)*

$\Longrightarrow$ Diagrammatism. . .

# Conclusions

Peirce (with O.H. Mitchell) and Frege invented quantifiers as variable-binding operators independently, Frege in 1879 (*Begriffsschrift*) and Peirce in 1883-5 (*Studies in Logic*, Logic of Relatives, improved in the 1885 paper).
[Peirce led the Metaphysical Club at the Johns Hopkins beginning 1879 so the year 1879 may indeed be commemorated. . . ]

- But unlike Frege, quantification in the logic of relatives is over some freely chosen domain of objects of discourse, not over the 'absolute totality' of all objects. Thus Peirce initiated the model-theoretic idea of isolating well-defined parts of 'the world' against which to evaluate truth of formulas ('true in a model').
- Peirce did not, unlike Frege, define inductively the wffs of a formal language. All the definitions were *explicit*.

# Conclusions

- Everyone adopted Peirce's $(\Pi, \Sigma)$ or Peano's $(\forall, \exists)$ notation for quantifiers until about 1910, when Russell decided to promote Frege's cumbersome alternative, which others found incomprehensible.
- Russell changed the term 'Peirce–Peano logic', already known to refer to first-order logic, to 'Peano–Frege logic', later to be called the 'Frege–Russell' logic. . .
- Much later, in 1946, Russell confessed, "I am, I confess to my shame, an illustration of the undue neglect from which Peirce has suffered in Europe. . . ".

# Part I

# Lecture 3: Some Logical Developments

# Propositional Logic

In the same 1885 paper, Peirce applied the concept of truth-values to show that (quantifier-free) formulas are *valid* (necessarily, i.e. logically true). He also delineates a propositional subsystem of first-order logic.

> *To find whether a formula is necessarily true substitute* **f** *[falsum] and* **v** *[verum] for the letters and see whether it can be supposed false by any such assignment of values. (CP 3.387)*

Since validity of $\varphi$ is truth in all interpretations (assignments of values to atomic propositions of $\varphi$), one can show that $\varphi$ is valid by assuming that $\varphi$ is false and show that this leads to a contradiction (Semantic Tableaux method, Hintikka & Beth 1955).

# Standard Truth Tables

Peirce introduced these in 1880.

| $\wedge$ | $v$ | $f$ |
|---|---|---|
| $v$ | $v$ | $f$ |
| $f$ | $f$ | $f$ |

(conjunction, $\varphi \wedge \psi$)

| $\vee$ | $v$ | $f$ |
|---|---|---|
| $v$ | $v$ | $v$ |
| $f$ | $v$ | $f$ |

(disjunction, $\varphi \vee \psi$)

| $\neg$ | |
|---|---|
| $v$ | $f$ |
| $f$ | $v$ |

(negation, $\neg\varphi$)

Since $P \rightarrow Q$ is equivalent to $\neg P \vee Q$, it is true if $P = f$ or $Q = v$, and false only if $P = v$ and $Q = f$.

# Validity (1885)

For example, to prove the validity of

$$(P \to Q) \to ((Q \to R) \to (P \to R)), \tag{15}$$

assume (15) is false, and show a derivation of a contradiction:

| | |
|---|---|
| (1) | $(P \to Q) \to ((Q \to R) \to (P \to R)) = f$ |
| (2) | $(P \to Q) = v$ (from 1) |
| (3) | $((Q \to R) \to (P \to R)) = f$ (from 1) |
| (4) | $(Q \to R) = v$ (from 3) |
| (5) | $(P \to R) = f$ (from 3) |
| (6) | $P = v$ (from 5) |
| (7) | $R = f$ (from 5) |
| (8) | $(v \to Q) = v$ (from 2 and 6) |
| (9) | $(Q \to f) = v$ (from 4 and 7) |
| (10) | $Q = v$ (from 8) |
| (11) | $Q = f$ (from 9) |

So there is no 'state of things' in which (15) would be false, so it must be logically true.

# Logical NAND and NOR Operators (1880)

These are two-place connectives with a special property: they suffice alone to characterise a truth-functionally complete system of propositional logic (= all other connectives can be defined by their means).

**Peirce's Arrow** (NOR, from 'not or'): $P \downarrow Q$, is true iff $\neg(P \vee Q)$ is true. ($P \downarrow Q$: *neither P nor Q*).

$$\begin{array}{c|cc} \downarrow & v & f \\ \hline v & f & f \\ f & f & v \end{array} \qquad \left(\begin{array}{c|cc} \vee & v & f \\ \hline v & v & v \\ f & v & f \end{array}\right)$$

$$\begin{aligned} &\neg P ::= P \downarrow P \\ &P \vee Q ::= (P \downarrow Q) \downarrow (P \downarrow Q) \\ &P \wedge Q ::= (P \downarrow P) \downarrow (Q \downarrow Q) \\ &P \rightarrow Q ::= ((P \downarrow Q) \downarrow Q) \downarrow ((P \downarrow Q) \downarrow Q). \end{aligned}$$

# Logical NAND and NOR Operators (1880)

**Sheffer Stroke** (NAND, from 'not and'): in symbols $P \mid Q$, is true iff $\neg(P \wedge Q)$ is true.

$$\begin{array}{c|cc} \mid & v & f \\ \hline v & f & v \\ f & v & v \end{array} \qquad \left(\begin{array}{c|cc} \wedge & v & f \\ \hline v & v & f \\ f & f & f \end{array}\right)$$

$$\begin{aligned} &\neg P ::= P \mid P \\ &P \wedge Q ::= (P \mid Q) \mid (P \mid Q) \\ &P \vee Q ::= (P \mid P) \mid (Q \mid Q) \\ &P \rightarrow Q ::= (P \mid Q) \mid P. \end{aligned}$$

# Logical NAND and NOR Operators (1880)

In 1886, Peirce suggested to Allan Marquand, who had designed some mechanical logic machines for syllogistic reasoning, that

> *it is by no means hopeless to make a machine for really difficult mathematical problems. But you would have to proceed step by step. I think electricity would be the best thing to rely on.*

He then showed how switching circuits can be connected serially and in parallel, noting that these two configurations correspond to multiplication (sum $\sim$ disjunction) and addition (product $\sim$ conjunction) in logic.

Nowadays circuits are typically composed out of NAND and NOR operations. But it was not only until after Claude Shannon's (1937) suggestions that such a machine that Peirce proposed was eventually constructed.

# The Origins

The invention of three-valued logic is typically attributed to Emil Post and Jan Łukasiewicz, both in 1920.

However, in the unpublished *Logic Notebook* (MS 339, 1909), Peirce developed semantics for three-valued logic ("Triadic Logic").

These notes were first studies and published by Fisch & Turquette in 1966.

Peirce's purpose was to include within the study of logic also propositions which are neither true (v, *verum*) nor false (f, *falsum*), using truth tables. He termed them the *limit* propositions, denoted by $l$.

# Peirce's Triadic Logic

As noted in relation to the principle of excluded middle (Lecture 2), Peirce thought that there are some 'limit' propositions neither true nor false:

> *[Triadic logic], though not rejecting entirely the Principle of Excluded Middle, nevertheless recognizes that every proposition, $S$ is $P$, is either true, or false, or else $S$ has a lower mode of being such that it can neither be determinately $P$, nor determinately not $P$, but is at the limit between $P$ and not $P$. (MS 339)*

He defined several connectives that may realise that idea.

# Peirce's Triadic Logic

He proposed four one-place connectives:

| $\varphi$ | $\bar{\varphi}$ | $\bar{\varphi}^o$ | $\grave{\varphi}$ | $\acute{\varphi}$ |
|---|---|---|---|---|
| $v$ | $f$ | $l$ | $f$ | $l$ |
| $l$ | $l$ | $l$ | $v$ | $f$ |
| $f$ | $v$ | $l$ | $l$ | $v$ |

| $\neg$ | |
|---|---|
| $v$ | $f$ |
| $f$ | $v$ |

(‘classical’ negation $\neg\varphi$)

- Connective $\bar{\varphi}$ corresponds to *strong negation* $\sim\varphi$.
- Connective $\bar{\varphi}^o$ corresponds to *tertium function* $T(\varphi)$.
- Connectives $\grave{\varphi}$ and $\acute{\varphi}$ correspond to *Post negations* $-\varphi$ and $—\varphi$.

# Peirce's Triadic Logic

He proposed six two-place connectives:

$$\begin{array}{c|ccc} \mathbf{\Theta} & v & l & f \\ \hline v & v & v & v \\ l & v & l & l \\ f & v & l & f \end{array} \qquad \left( \begin{array}{c|cc} \vee & v & f \\ \hline v & v & v \\ f & v & f \end{array} \right)$$

Like disjunction $\vee$, but in case one or both of the 'juncts' being indeterminate and the other false, the proposition is indeterminate.

$$\begin{array}{c|ccc} \mathbf{Z} & v & l & f \\ \hline v & v & l & f \\ l & l & l & f \\ f & f & f & f \end{array} \qquad \left( \begin{array}{c|cc} \wedge & v & f \\ \hline v & v & f \\ f & f & f \end{array} \right)$$

Like conjunction $\wedge$, but in case one or both of the 'juncts' being indeterminate and the other true, the proposition is indeterminate.

# Peirce's Triadic Logic

| **Y** | $v$ | $l$ | $f$ |
|---|---|---|---|
| $v$ | $v$ | $l$ | $v$ |
| $l$ | $l$ | $l$ | $l$ |
| $f$ | $v$ | $l$ | $f$ |

$$\left(\begin{array}{c|cc} \vee & v & f \\ \hline v & v & v \\ f & v & f \end{array}\right)$$

Like disjunction, but in case one or both of the 'juncts' being indeterminate the proposition is indeterminate.

| **Ω** | $v$ | $l$ | $f$ |
|---|---|---|---|
| $v$ | $v$ | $l$ | $f$ |
| $l$ | $l$ | $l$ | $l$ |
| $f$ | $f$ | $l$ | $f$ |

$$\left(\begin{array}{c|cc} \wedge & v & f \\ \hline v & v & f \\ f & f & f \end{array}\right)$$

Like conjunction, but in case one or both of the 'juncts' being indeterminate the proposition is indeterminate.

# Peirce's Triadic Logic

| $\mathbf{\Phi}$ | $v$ | $l$ | $f$ |
|---|---|---|---|
| $v$ | $v$ | $v$ | $v$ |
| $l$ | $v$ | $l$ | $f$ |
| $f$ | $v$ | $f$ | $f$ |

$$\left(\begin{array}{c|cc} \vee & v & f \\ \hline v & v & v \\ f & v & f \end{array}\right)$$

Like disjunction, but the proposition is indeterminate only if both 'juncts' are indeterminate. Elsewhere indeterminacy is irrelevant.

| $\mathbf{\Psi}$ | $v$ | $l$ | $f$ |
|---|---|---|---|
| $v$ | $v$ | $v$ | $f$ |
| $l$ | $v$ | $l$ | $f$ |
| $f$ | $f$ | $f$ | $f$ |

$$\left(\begin{array}{c|cc} \wedge & v & f \\ \hline v & v & f \\ f & f & f \end{array}\right)$$

Like conjunction, but the proposition is indeterminate only if both 'juncts' are indeterminate. Elsewhere indeterminacy is irrelevant.

'Logic of ordinary conversation'.

# Peirce's Triadic Logic

Peirce thus rejected the *Principle of Bivalence* (PB), according to which for every proposition $\varphi$, if $\varphi$ is not true then $\varphi$ is false, and if $\varphi$ is not false then $\varphi$ is true.

The propositions to which the PB does not apply are those that lie in the 'narrow boundary area' between true and false.

By taking suitable two-connective systems, say those of $\{\boldsymbol{\Psi}, \acute{\ }\}$ or $\{\mathbf{Z}, \grave{\ }\}$, we have a *functionally complete* set of connectives.

# Conclusions

It is commonly held that Peirce made two contributions to logic that were his most important ones:

1. The Algebraic Logic of Relations, and
2. The Theory of Quantification, stimulated by the developments in the algebraic theory.

These emerged during the 'first phase' (1870–1885), so we should add from the 'later phase' (1909) at least his

3. Triadic Logic.

In between, there was ample time for yet another major contribution, namely his theory of

4. Diagrammatic Logics (1895–).

Peirce himself though that to have been his "chef d'oeuvre" in logic. We turn to that in the couple of next lectures.

# Conclusions

1. The general significance of these logical innovations was that they marked the milestones in the development of the *model-theoretic* and *semantic* traditions in logic.
2. A founder of modern logic in giving us truth-tables, proof methods, first-order logic, and a lot more.
3. Did not rest content with having only propositions that are either **true** or **false**, as some indeterminate ones could be **neither**.

# Part II

## Lecture 4: Existential Graphs, System ALPHA

# Peirce's Comments

*I do not think I ever* **reflect** *in words: I employ visual diagrams, firstly, because this way of thinking is my natural language of self-communion, and secondly, because I am convinced that it is the best system for the purpose. (MS 619, 1909).*

Peirce's goal was a logical analysis of thought and reasoning that is rigorous and valid also when symbolic expressions fall short of fulfilling that purpose.

> *There are countless Objects of consciousness that words cannot express; such as the feelings a symphony inspires or that which is in the soul of a furiously angry man in [the] presence of his enemy. (MS 499, 1906).*

No one has invented logical diagrams for feelings, but Peirce strongly believed in their plausibility.

# "Moving Pictures of Thought"

According to Peirce, graphical representation of natural language puts before us

- "moving pictures of thought" (CP 4.11)
- "a moving picture of the action of the mind in thought" (MS 298: 1)
- "a portraiture of Thought".

The precise vehicle is the iconic logic of diagrams, which will

- "furnish a moving picture of the intellect" (MS 298: 10) and provides a "system for diagrammatizing intellectual cognition" (MS 292: 41).

# The Turn to the Iconic

Mathematics is iconic:

> *Logic may be defined as the science of the laws of the stable establishment of beliefs. Then,* **exact** *logic will be that doctrine of the conditions of establishment of stable belief which rests upon perfectly undoubted observations and upon mathematical, that is, upon* **diagrammatical**, *or,* **iconic**, *thought. (CP 3.429, 1896)*

What is iconic thought?

> *We form in the imagination some sort of diagrammatic, that is,* **iconic**, *representation of the facts, as skeletonized as possible. The impression of the present writer is that with ordinary persons this is always a visual image, or mixed visual and muscular; but this is an opinion not founded on any systematic examination. If visual, it will either be* **geometrical**, *that is, such that familiar spatial relations stand for the relations asserted in the premisses, or it will be* **algebraical**, *where the relations are expressed by objects which are imagined to be subject to certain rules, whether conventional or experiential. (CP 2.778, 1901)*

# Icon, Index, Symbol

According to Peirce, three kinds of signs are necessary in logic.

> *The first is the diagrammatic sign or* **icon**, *which exhibits a similarity or analogy to the subject of discourse; the second is the* **index**, *which like a pronoun demonstrative or relative, forces the attention to the particular object intended without describing it; the third [or* **symbol**] *is the general name or description which signifies its object by means of an association of ideas or habitual connection between the name and the character signified. (CP 1.369)*

Iconicity comes in many guises:

> *Every picture (however conventional its method) is essentially a representation of that [iconic] kind. So is every diagram, even although there be no sensuous resemblance between it and its object, but only an analogy between the relations of the parts of each. (CP 2.279).*

# Diagrammatic Logic

What is essential here?

- *Iconic* representations: denote things represented by likeness, semblance, analogy (*graphs*, *diagrams*, *models*, *sets of sentences*, . . . ).
- May be abstract, structural, intellectual likeness ('true-in-a-model', (homo/auto)morphisms, structure-preserving maps, . . . )
- Modern incarnation: Conceptual Graphs (CG) in Computer Science & AI
- The original formulation was in terms of the very expressive system of *Existential Graphs* (EGs, 1896).

# Image, Diagram, Metaphor

> *A* **Diagram** *is a [sign] which is predominantly an icon of relations and is aided to be so by conventions. Indices are also more or less used. (MS 492: 22).*

A diagram should be "as iconic as possible" in order to represent "visible relations" (MS 492: 22). Nowadays there are the heterogeneous logics that are not fully iconic.

Not all iconicity is diagrams, however. Iconic signs (*hypoicons*) fall into three classes:

> *Those which partake of simple qualities, or First Firstnesses, are* **images***; those which represent the relations, mainly dyadic, or so regarded, of the parts of one thing by analogous relations in their own parts, are* **diagrams***; those which represent the representative character of a representamen by representing a parallelism in something else, are* **metaphors***. (EP 2:273, 1903).*

# Graph, Logical Graph, Existential Graph

*A* **graph** *is a superficial diagram composed of the* **sheet** *[Sheet of Assertion] upon which it is written or drawn, of* **spots** *or their equivalents, of* **lines of connection**, *and (if need be) of* **enclosures**. *The type, which it is supposed more or less to resemble, is the structural formula of the chemist.*

*A* **logical graph** *is a graph representing logical relations iconically, so as to be an aid to logical analysis.*

*An* **existential graph** *is a logical graph governed by a system of representation founded upon the idea that the sheet upon which it is written, as well as every portion of that sheet, represents one recognized universe, real or fictive, and that every graph drawn on that sheet, and not cut off from the main body of it by an enclosure, represents some fact existing in that universe, and represents it independently of the representation of another such fact by any other graph written upon another part of the sheet, these graphs, however, forming one composite graph. (CP 4.419–21).*

# Schematically:

**idea**

$\Uparrow$

*Vorstellung*

$\Uparrow$

**representation**

$\Uparrow$

**Icon** $\Leftarrow$ Index $\Leftarrow$ Symbol

$\Uparrow$

Image $\Leftarrow$ **Diagram** $\Leftarrow$ Metaphor

$\Uparrow$ $\Uparrow$ $\Uparrow$

? Graph ?

$\Uparrow$

Logical Graph

$\Uparrow$

Existential Graph (Alpha, Beta, Gamma)

*I extend logic to embrace all the necessary principles of semeiotic, and I recognize a logic of icons, and a logic of indices, as well as a logic of symbols. (CP 4.9, 1906).*

# Alpha, Beta, Gamma

1. Alpha Graphs $\sim$ propositional logic
   *Cat and dog are on a mat.*
2. Beta Graphs $\sim$ predicate logic
   *Every man is mortal.*
3. Gamma Graphs $\sim$
   3.1 Modal logic;
   *It is possible that it rains.*
   3.2 Higher-order logic;
   *Aristotle has all the virtues of a philosopher.*
   3.3 Metagraphs;
   *'You are a good goalkeeper' is much to be said.*
   3.4 Non-declaratives;
   interrogatives, imperatives, emotions, interpretation of music. . .

# Alpha Graphs

## Definition

The set of Alpha Graphs $\mathcal{G}_\alpha$ is the smallest set of satisfying:

1. *Sheet of Assertion* (SA): $\boxed{\quad \text{G1}}$ $\in \mathcal{G}_\alpha$.
2. Closure under *juxtaposition*:

   If $P_1 \in \mathcal{G}_\alpha$, $P_2 \in \mathcal{G}_\alpha$ $\ldots P_n \in \mathcal{G}_\alpha$, then $\boxed{P_1 \ldots P_n}$ $\in \mathcal{G}_\alpha$.
3. Closure under *cuts*:

   If $P_1 \in \mathcal{G}_\alpha$, then $\boxed{\quad (P_1)}$ $\in \mathcal{G}_\alpha$.

## Remark

- Cuts may not overlap.
- Juxtaposition is *commutative* and *associative*.

# Alpha Graphs

EGs are scribed on a surface.

1. **Sheet of Assertion**

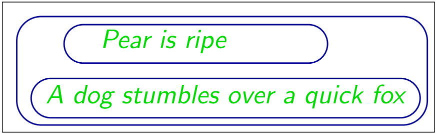
$\top$ (verum)

2. **Juxtaposition** $\sim$ conjunction

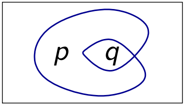

$p \wedge q$

3. **Cut** $\sim$ negation


$\neg p$

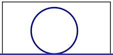



$\neg(\neg p \wedge \neg q) = p \vee q$




(the scroll)

$\neg(p \wedge \neg q) = \neg p \vee q = p \rightarrow q$



$\neg\top = \bot$ (falsum)

# Alpha Graphs

## Definition (Area)

Space within the cut without the cut is the *area of the cut*.

## Definition (Enclosure)

A cut, its area and everything in that area comprise the *enclosure of the cut*.

## Remark

1. Area is not part of the SA.
2. Area is not a graph.
3. Cut is not a graph.
4. Enclosure is a graph.

# Alpha Graphs

## Definition (Positive and negative areas)

1.a Any graph $P \in \mathcal{G}_\alpha$ not enclosed by any cut or enclosed by an even number of cuts is *evenly enclosed*.

b Any graph $P \in \mathcal{G}_\alpha$ enclosed by an odd number of cuts is *oddly enclosed*.

2.a Area on which an evenly enclosed graph rests is *positive*.

b Area on which an oddly enclosed graph rests is *negative*.

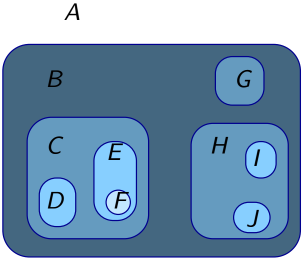
## Remark

The union of evenly and oddly enclosed graphs of any $P \in \mathcal{G}_\alpha$ comprise the set of all subgraphs of $P$.

# Alpha Graphs

## Definition (Nest)

- A linearly ordered finite sequence of areas from the SA to the areas of cuts of increasing depth makes a *nest*.
- A nest terminating on a cut-free area is a *maximal nest*.




1. One of 5 areas, or 4 cuts A-B-C-E-F

Three of 4 areas or 3 cuts each:

2. A-B-C-D
3. A-B-H-I
4. A-B-H-J
5. One of 3 areas, or 2 cuts, A-B-G.

# The Sheet of Assertion

In Peirce's own terms ('Conventions I–VI' (I–III ALPHA, IV–VI BETA).

**Convention No. I:** The Sheet of Assertion (SA) is

> *considered as representing the universe of discourse, and as asserting whatever is taken for granted between the* **graphist** *[***utterer***] and the* **interpreter** *[***grapheus***] to be true of that universe. The sheet of assertion is, therefore, a graph. (CP 4.396).*

SA is

> *a surface upon which the* **utterer** *and* **interpreter** *will, by force of a voluntarily and actually contracted* **habit***, recognize that whatever is scribed upon it and is interpretable as an assertion is to be recognized as an assertion, although it may refer to a mere* **idea** *as its subject.*

# Entire and Partial Graphs (Juxtaposition)

**Convention No. II:** Entire and Partial Graphs:

> *The graph which consists of all the graphs on the sheet of assertion, or which consists of all that are on any one area severed from the sheet, shall be termed the* **entire** *graph of the sheet of assertion or of that area, as the case may be. Any part of the entire graph which is itself a graph shall be termed a* **partial** *graph of the sheet or of the area on which it is. (CP 4.398).*

Any two partial graphs of an entire graph express juxtaposition, in other words (commutative and associative) conjunction between them.

# Cuts

**Convention No. III:** Cuts:

*By a* **Cut** *shall be understood to mean a self-returning linear separation (naturally represented by a fine-drawn or peculiarly colored line) which severs all that it encloses from the sheet of assertion on which it stands itself, or from any other area on which it stands itself.*

*— The whole space within the cut (but not comprising the cut itself) shall be termed the* **area** *of the cut. Though the area of the cut is no part of the sheet of assertion, yet the cut together with its area and all that is on it, conceived as so severed from the sheet, shall, under the name of the* **enclosure** *of the cut, be considered as on the sheet of assertion or as on such other area as the cut may stand upon.*

*— Two cuts cannot intersect one another, but a cut may exist on any area whatever. Any graph which is unenclosed or is enclosed within an* **even number of cuts** *shall be said to be* **evenly enclosed***; and any graph which is within an* **odd number of cuts** *shall be said to be* **oddly enclosed***.*

*— A cut is not a graph; but an enclosure is a graph. The sheet or other area on which a cut stands shall be called the* **place** *of the cut. (CP 4.399).*

# Scrolls

A special case of cuts is a scroll:

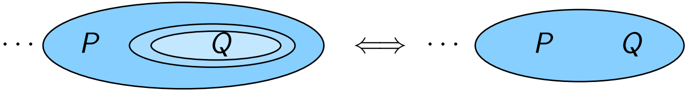
> *A pair of cuts, one within the other but not within any other cut that that other is not within, shall be called a* **scroll**. *The outer cut of the pair shall be called the* **outloop**, *the inner cut the* **inloop**, *of the scroll. The area of the inloop shall be termed the* **inner close** *of the scroll; the area of the outloop, excluding the enclosure of the inloop (and not merely its area), shall be termed the* **outer close** *of the scroll. (CP 4.400).*

The scroll is the iconic counterpart of material implication.

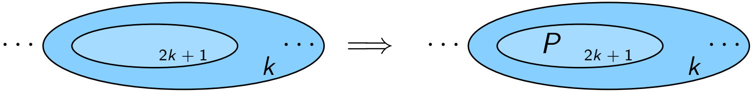
# Proofs

Five rules of transformation:

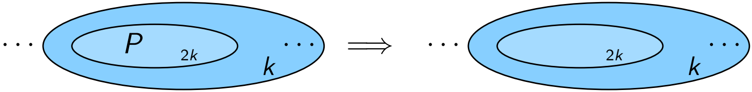
1. **Add/remove double cuts**:




2. **Insertion**: Any $P \in \mathcal{G}_\alpha$ may be added on negative area:

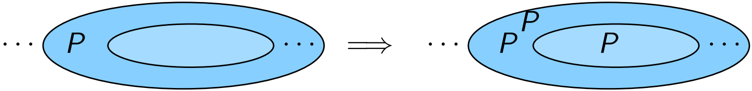



3. **Erasure**: Any $P \in \mathcal{G}_\alpha$ may be erased from positive area:

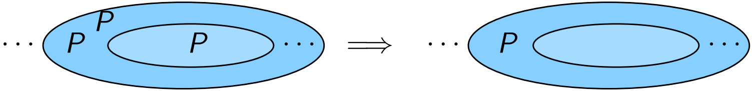


# Proofs

4. **Iteration**: Any copy of $P$ may be scribed on the same area or on the area in its nest (not part of $P$):




5. **Deiteration**: Any copy of $P$ may be removed from the same area or from the area in its nest (not part of $P$):



# Next Topics

Lecture 5:

- BETA Graphs
- Semantics for EGs.

Lecture 5:

- GAMMA Graphs (modalities etc.)

And the story continues with

- Continuity and
- Pragmaticism, and their relationships with EGs...